Duality and the geometric measure of entanglement of general multiqubit *W* states

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We find the nearest product states for arbitrary generalized W states of n qubits, and show that the nearest product state is essentially unique if the W state is highly entangled. It is specified by a unit vector in Euclidean n-dimensional space. We use this duality between unit vectors and highly entangled W states to find the geometric measure of entanglement of such states.

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I. INTRODUCTION

The quantification of entanglement of multipartite pure states presents a real challenge to physicists. Intensive studies are under way, and different entanglement measures have been proposed over the years [1–6]. However, it is generally impossible to calculate their value because the definition of any multipartite entanglement measure usually includes a massive optimization over certain quantum protocols or states [7–9].

Inextricable difficulties of the optimization are rooted in a tangle of different obstacles. First, the number of entanglement parameters grows exponentially with the number of particles involved [10]. Second, in the multipartite setting several inequivalent classes of entanglement exist [11,12]. Third, the geometry of entangled regions of robust states is complicated [13]. All of these factors make the usual optimization methods ineffective [13–15]. Concise and elegant tools are required to overcome this problem.

A widely used measure for multipartite systems is the geometric measure of entanglement E_g [16], that is, the distance from the nearest product state. For an n-part pure state ψ, it is defined as $E_g(\psi) = -2\ln g(\psi)$, where the maximal product overlap $g(\psi)$ is given by

$$g(\psi) = \max_{u_1,u_2,\dots,u_k} |\langle\psi|u_1u_2\cdots u_k\rangle|,$$

and the maximization is performed over all product states. The maximal product overlap has many remarkable applications. Among them are the following. It singles out the multipartite states applicable for perfect quantum teleportation and superdense coding [13], it can create a generalized Schmidt decomposition for arbitrary n-part systems [17], it identifies irregularity in channel capacity additivity [18], it quantifies the difficulty of distinguishing multipartite quantum states by local means [19], it is a good entanglement estimator for quantum phase transitions in spin models [20], it detects a one-parameter family of maximally entangled states [21], and it can be easily estimated in experiments [22].

In what follows states with $g^2 > 1/2$ are referred to as slightly entangled, states with $g^2 < 1/2$ are referred to as highly entangled, and states with $g^2 = 1/2$ are referred to as shared quantum states. In this paper we show how to calculate the maximal product overlap of an arbitrary W state [11]. The method is to establish a one-to-one correspondence between highly entangled W states and their nearest product states.

Consider first generalized Greenberger-Horne-Zeilinger (GHZ) states [23], that is, states that can be written $|\mathrm{GHZ}\rangle = a|0\cdots0\rangle + b|1\cdots1\rangle$ in some product basis. Such states are fragile under local decoherence, that is, they become disentangled by the loss of any one party, and they are not highly entangled in the sense defined above. The geometric measure of these states is computed easily since the maximal overlap simply takes the value of the modulus of the larger coefficient $|a|$ or $|b|$ [24]. Accordingly, the nearest separable state is the product state with the larger coefficient. Thus many generalized GHZ states with different maximal overlaps can have the same nearest product state.

Consider now generalized W states [25], which can be written

$$|W_n\rangle = c_1|100\cdots0\rangle + c_2|010\cdots0\rangle + \cdots + c_n|00\cdots01\rangle. \tag{1}$$

Without loss of generality, we consider only the case of positive parameters c_k, since the phases of the coefficients c_k can be eliminated by redefinitions of local states $|1_k\rangle$, $k = 1,2,\dots,n$. The states (1) are robust against decoherence [26], that is, loss of any $n-2$ parties still leaves them in a bipartite entangled state. Surprisingly, if the state is slightly entangled, then we have the same situation as for generalized GHZ states: the maximal overlap is the largest coefficient and, as before, many states can have the same nearest product state [27]. However, the situation is changed drastically when the state is highly entangled. The calculation of the maximal overlap in this case is a very difficult problem, and the maximization has been performed only for relatively simple systems [9,14,16,24,27–30].

On the other hand, different highly entangled W states have different nearest product states. This makes it possible to map the W state to its nearest product state and quickly obtain its geometric measure of entanglement. More precisely, we construct two bijections. The first one creates a map between highly entangled n-qubit W states and n-dimensional unit vectors $\mathbf{x}$. The second one does the same between n-dimensional unit vectors and n-part product states. Thus we obtain a double map, or *duality*, as follows:

$$|W_n\rangle \ \leftrightarrow \ \mathbf{x} \ \leftrightarrow \ |u_1\rangle \otimes |u_2\rangle \otimes \cdots \otimes |u_n\rangle. \tag{2}$$

The main advantage of the map is that, if one knows any of the three vectors, then one instantly finds the other two.

II. CLASSIFYING MAP

Now we prove a theorem that provides a basis for the map.

Theorem 1. Let $|W_n\rangle$ be an arbitrary W state (1) with non-negative real coefficients c_i, and let $|u_1\rangle \otimes |u_2\rangle \otimes \cdots \otimes |u_n\rangle$ be its nearest product state. Then the phase of $|u_k\rangle$ can be chosen so that

$$|u_k\rangle = \sin\theta_k|0\rangle + \cos\theta_k|1\rangle, \quad 0 \leqslant \theta_k \leqslant \frac{\pi}{2}, \quad k = 1,2,\ldots,n,$$

where

$$\cos^2\theta_1 + \cos^2\theta_2 + \cdots + \cos^2\theta_n = 1. \tag{3}$$

Proof. The nearest product state is a stationary point for the overlap with $|W_n\rangle$, so the states $|u_k\rangle$ satisfy the nonlinear eigenvalue equations [9,16,17]

$$\langle u_1 u_2 \cdots \widehat{u_k} \cdots u_n | \mathrm{W}_n\rangle = g|u_k\rangle, \quad k = 1,2,\ldots,n, \tag{4}$$

where the caret means exclusion. We can choose the phase of $|u_k\rangle$ so that $|u_k\rangle = \sin\theta_k|0\rangle + e^{i\phi_k}\cos\theta_k|1\rangle$, and then (4) gives the pair of equations

$$c_k \prod_{j\neq k} \sin\theta_j = g e^{i\phi_k}\cos\theta_k, \tag{5a}$$

$$\sum_{l\neq k} e^{-i\phi_l} c_l \cos\theta_l \prod_{j\neq k,l} \sin\theta_j = g\sin\theta_k. \tag{5b}$$

Equation (5a) shows that $ge^{i\phi_k}$ is real, so $\phi_k = -\arg(g)$ is independent of k. Then the modulus of the overlap $|\langle u_1 \cdots u_n|W_n\rangle|$ is independent of ϕ, so we can assume that $\phi = 0$. Now multiplication of Eq. (5b) by $\sin\theta_k$ and use of Eq. (5a) gives Eq. (3). Thus the angles $\cos\theta_k$ define a unit n-dimensional Euclidean vector $\mathbf{x}$. We can also define a length r as follows. From Eq. (5a) it follows that the ratio $\sin 2\theta_k/c_k$ does not depend on k. If this ratio is nonzero, we can define

$$\frac{1}{r} \equiv \frac{\sin 2\theta_1}{c_1} = \frac{\sin 2\theta_2}{c_2} = \cdots = \frac{\sin 2\theta_n}{c_n}. \tag{6}$$

III. HIGHLY ENTANGLED *W* STATES

Equations (5) admit a trivial solution $\sin 2\theta_k = 0, k = 1,2,\ldots,n$ and a special solution with nonzero values of all sines. The trivial solution gives the largest coefficient of $|W_n\rangle$ for the maximal overlap and is valid for slightly entangled states. We consider them later and now focus on the special solutions. From Eq. (6) it follows that

$$\cos^2\theta_k = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{c_k^2}{r^2}}\right), \quad k = 1,2,\ldots,n. \tag{7}$$

The plus sign means that $\cos 2\theta_k > 0$. Then from Eq. (3) it follows that this is possible for at most one angle; specifically, we prove that, if $\cos 2\theta_k > 0$ for some k, then c_k is the largest coefficient in Eq. (1). Suppose $\cos 2\theta_k > 0$ but c_k is not the largest coefficient and there exists a greater coefficient, say c_l. Then from Eq. (6) it follows that $\sin 2\theta_l > \sin 2\theta_k > 0$ and consequently $|\cos 2\theta_l| < |\cos 2\theta_k|$. Now we rewrite Eq. (3) as follows:

$$-\cos 2\theta_1 - \cos 2\theta_2 - \cdots - \cos 2\theta_n = n - 2. \tag{8}$$

From $|\cos 2\theta_l| < |\cos 2\theta_k|$ and $\cos 2\theta_k > 0$ it follows that $-\cos 2\theta_k - \cos 2\theta_l < 0$, which is in contradiction with Eq. (8). Thus c_k must be the largest coefficient.

Without loss of generality, we assume that $0 \leqslant c_1 \leqslant \cdots \leqslant c_n$. Then in (7) we must take the $-$ sign for $k = 1,\ldots,n-1$ and (3) becomes

$$\sqrt{1 - \frac{c_1^2}{r^2}} + \cdots + \sqrt{1 - \frac{c_{n-1}^2}{r^2}} \pm \sqrt{1 - \frac{c_n^2}{r^2}} = n - 2. \tag{9}$$

We will denote the left-hand sides of these equations as $f_\pm(r)$. We also use $f_0(r)$ to denote this expression without the last term. The function $r(c_1,c_2,\ldots,c_n)$ defined by $f_+(r) = n-2$ is a completely symmetric function of the state parameters c_k. In contrast, the function defined by $f_-(r) = n-2$ is an asymmetric function since its dependence on the maximal coefficient c_n is different. Thus in Eq. (9) the upper and lower signs describe symmetric and asymmetric entangled regions of highly entangled states, respectively.

For highly entangled states, Eqs. $(9)_\pm$ uniquely define r as a function of the state parameters c_k. More precisely, we have the following theorem.

Theorem 2. There are two critical values r_1 and r_2 of the largest coefficient c_n, that is, functions of $c_1,\ldots,c_{n-1}$ such that

1. if $c_n \leqslant r_1$, there is a unique solution of $(9)_+$ and no solution of $(9)_-$;
2. if $c_n = r_1$, both $(9)_+$ and $(9)_-$ have a unique solution, the same for both;
3. if $r_1 < c_n \leqslant r_2$, there is no solution of $(9)_+$ and a unique solution of $(9)_-$;
4. if $c_n > r_2$, neither $(9)_+$ nor $(9)_-$ has a solution. In this case the state $|\mathrm{W}_n\rangle$ is slightly entangled.

The value r_1 is the solution of $f_0(r_1) = n-2$, which exists and is unique since $f_0(c_{n-1}) < n-2$ and $f_0(r) \to n-1$ monotonically as $r \to \infty$; and r_2 is defined by

$$r_2^2 = c_1^2 + \cdots + c_{n-1}^2. \tag{10}$$

Then $r_2 \geqslant r_1$, for $f_0(r_2) \geqslant n-2 = f_0(r_1)$ using $\sqrt{x} \geqslant x$ for $0 \leqslant x \leqslant 1$. Since f_0 is an increasing function of r, it follows that $r_2 \geqslant r_1$. Now the theorem follows from the following properties of the functions $f_\pm(r)$ (f'_- is the derivative of f_-):

1. f_0 and f_+ are monotonically increasing functions of r.
2. $f_+(r) \to n$ as $r \to \infty$.
3. If $c_n \leqslant r_1$, $f_+(c_n) = f_0(c_n) \leqslant f_0(r_1) = n-2$.
4. If $c_n \geqslant r_1$, then $f_+(r) \geqslant n-2$ for all $r > r_1$.
5. If $c_n < r_1$, then $f_-(c_n) < n-2$.
6. If $c_n > r_1$, then $f_-(c_n) > n-2$.
7. If $c_n < r_2$, then $f_-(r) < n-2$ for large r.
8. If $c_n > r_2$ then $f_-(r) > n-2$ for large r.

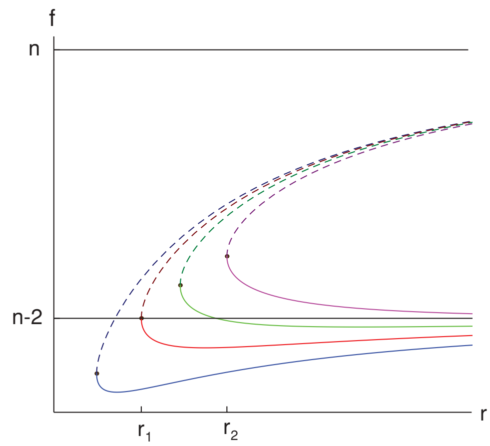

FIG. 1. (Color online) Behavior of the functions $f_\pm$ for five-qubit W states. The functions $f_+(r)$ (dotted line) and $f_-(r)$ (solid line) are plotted against r in the four cases $c_n < r_1$, $c_n = r_1$, $r_1 < c_n < r_2$, and $c_n = r_2$.

9. $f'_-(c_n + \epsilon) < 0$ for small ϵ.
10. If $c_n > r_2$, then $f'_-(r) < 0$ for all $r \geqslant c_n$.

These properties are illustrated in Fig. 1.

IV. GEOMETRIC MEASURE

We can now identify the nearest product state, and the largest product state overlap $g(|Ww_n\rangle)$, for any W state $|W_n\rangle$, as follows.

Theorem 3. If $c_n \geqslant 1/2$, the state $|W_n\rangle$ defined by (1) is slightly entangled. Its nearest product state is $|0\cdots 01\rangle$, with overlap $g(|W_n\rangle) = c_n$.

If $c_n \leqslant 1/2$, the state $|\mathrm{W}_n\rangle$ is highly entangled and has nearest product state

$$|u_1\rangle\cdots|u_n\rangle \quad \text{where } |u_k\rangle = \sin\theta_k|0\rangle + |e^{i\phi}\cos\theta_k|1\rangle, \tag{11}$$

with which its overlap is

$$g = 2r\sin\theta_1\sin\theta_2\cdots\sin\theta_n. \tag{12}$$

Here r is the solution of $(9)_\pm$, whose existence and uniqueness are guaranteed by Theorem 2; the phase ϕ is arbitrary; and θ_k is given by (7) with the $-$ sign for $k = 1,\ldots,n-1$, the $-$ sign for $k = n$ if r satisfies $(9)_+$, and the $+$ sign if r satisfies $(9)_-$.

Proof. The nonlinear eigenvalue equations (4) always have n solutions

$$g = c_k, \quad |u_i\rangle = \begin{cases} |0\rangle & \text{if} \quad i \neq k, \\ |1\rangle & \text{if} \quad i = k, \end{cases} \quad k = 1,\ldots,n.$$

If $c_n \geqslant 2$, that is, in case (4) of Theorem 2, there are no other stationary values, so the largest overlap $g(|W_n\rangle)$ equals the largest coefficient c_n, the corresponding product state being $|0\cdots 01\rangle$.

If $c_n < 1/2$ there is another stationary value given by (12). We will now show that this is larger than any of the trivial stationary values c_k. We use the following inequality: If $y_1,\ldots,y_n$ are real numbers lying between 0 and 1 and satisfying $y_1 + \cdots + y_n \leqslant 1$, then

$$(1-y_1)(1-y_2)\cdots(1-y_n) \geqslant 1 - y_1 - y_2 - \cdots - y_n. \tag{13}$$

This is readily proved by induction. We can apply (13) to $n-1$ terms of Eq. (3) to get

$$\begin{aligned}(1-\cos^2\theta_1)\cdots(1-\cos^2\theta_{n-1})\\ \geqslant 1-\cos^2\theta_1-\cdots-\cos^2\theta_{n-1}\end{aligned}$$

or

$$\sin^2\theta_1\sin^2\theta_2\sin^2\theta_{n-1} \geqslant \cos^2\theta_n. \tag{14}$$

Now from Eq. (5a) it follows that $g^2 \geqslant c_n^2$. Thus g is the maximal product overlap, and the nearest product state is $|u_1\rangle\cdots|u_n\rangle$.

Next we prove that, if $|W_n\rangle$ is normalized, then $g^2 < 1/2$. For this we need another inequality: If $y_1,\ldots,y_n$ are real numbers lying between 0 and 1, and satisfying $y_1 + \cdots + y_n = n-1$, then

$$y_1 + \cdots + y_n \geqslant y_1^2 + \cdots + y_n^2 + 2y_1y_2\cdots y_n. \tag{15}$$

This can also be proved by induction.

From (6), and using $c_1^2 + \cdots + c_n^2 = 1$, we find

$$r^2 = \frac{1}{\sin^2 2\theta_1 + \cdots + \sin^2 2\theta_n}. \tag{16}$$

Hence (12) gives

$$g^2 = \frac{y_1y_2\ldots y_n}{y_1(1-y_1)+\cdots y_n(1-y_n)} \tag{17}$$

where $y_k = \sin^2\theta_k$. But $y_1 + \cdots + y_n = n-1$, so the inequality (15) applies, and gives $g^2 \leqslant 1/2$.

Finally, we summarize the correspondence between highly entangled W states, their nearest product states, and unit vectors in $\mathbb{R}^n$.

Theorem 4. There is a 1:1 correspondence between highly entangled states $|W_n\rangle$ defined by (1), their nearest product states with real non-negative coefficients, and unit vectors $\mathbf{x} \in \mathbb{R}^n$ with $0 < x_k < 1/\sqrt{2}$ $(k = 1,\ldots,n-1)$, $0 < x_n < 1$.

Proof. By Theorem 3, $|W_n\rangle$ is highly entangled if and only if $c_n < 1/2$. If this is the case, Theorem II and (7) show that its nearest product state is of the form (11), where $\mathbf{x} = (\cos\theta_1,\ldots,\cos\theta_n)$ is a unit vector in $\mathbb{R}^n$ in the region stated. The angles θ_k are given in terms of the coefficients c_k by (6), in which r is a function of the coefficients which, by Theorem III, is uniquely defined. The nearest product states $|u_1\rangle|u_2\rangle\cdots|u_n\rangle$ are determined by these angles, up to a phase ϕ, by $|u_k\rangle = \sin\theta_k|0\rangle + e^{i\phi}\cos\theta_k|1\rangle$, so there is only one nearest product state with real non-negative coefficients, and only one unit vector $\mathbf{x}$, for each highly entangled state $|W_n\rangle$. Conversely, given a unit vector $\mathbf{x} = (\cos\theta_1,\ldots,\cos\theta_n)$, the quantity r is determined by (16), and then the coefficients $c_1,\ldots,c_n$ are determined by (6). Thus the correspondences (2) are bijections.

The equations $(9)_\pm$ cannot always be explicitly solved to give analytic expressions for r in terms of the coefficients c_k. However, in some cases, including all states for $n = 3$, explicit solutions can be obtained. Then the angles θ_k can be calculated from (6), and Eq. (12) gives a formula for the maximal product overlap $g(|W_n\rangle)$. This formula is valid unless any of the angles θ_k vanishes, and restores all known results for the maximal overlap of highly entangled W states. When $n = 3$, it coincides with the formula (31) in Ref. [9]. When $c_1 = c_2 = \cdots = c_n$ it coincides with the formula (52) in Ref. [24], and when $n = 4$ and $c_3 = c_4$ it coincides with the formula (37) derived in Ref. [27].

When $\max(c_1^2, c_2^2, \ldots, c_n^2) = r_2^2 = 1/2$, the two expressions for $g(|W_n\rangle)$ given in Theorem IV coincide; these states are shared quantum states. The nearest product states and maximal overlaps of shared states are given by the first case of Theorem IV, but also they appear as asymptotic limits of the second case. Indeed, at the limit $\theta_n \to 0$, we have

$$\lim_{\theta_n \to 0} 2r \sin\theta_n \to c_n, \quad \lim_{\theta_n \to 0} 2r \cos\theta_k \to c_k, \quad k \neq n. \tag{18}$$

Thus the angle θ_n vanishes and the length of the vector $\boldsymbol{r}$ goes to infinity, but their product has a finite limit. Substituting these limits into Eq. (3), one obtains $c_n^2 \to r_2^2$. Therefore entangled regions of highly and slightly entangled states are separated by the surface $c_n^2 = 1/2$; for states on the surface, $r \to \infty$. All of these states can be used as a quantum channel for perfect teleportation and superdense coding [13].

V. SUMMARY

We have constructed correspondences among W states, n-dimensional unit vectors, and separable pure states. The map reveals two critical values for quantum state parameters. The first critical value separates symmetric and asymmetric entangled regions of highly entangled states, while the second one separates highly and slightly entangled states. The method gives an explicit expressions for the geometric measure when the state allows analytical solutions; otherwise it expresses the entanglement as an implicit function of state parameters.

It should be noted that the bijection between W states and n-dimensional unit vectors is not related directly to the geometric measure of entanglement. Therefore it is possible to extend the method to other entanglement measures. To this end one creates an appropriate bijection between unit vectors and optimization points of an entanglement measure one wants to compute. This work is in progress.

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