Bull. Korean Math. Soc. **49** (2012), No. 1, pp. 165–174
http://dx.doi.org/10.4134/BKMS.2012.49.1.165

ON A p-ADIC ANALOGUE OF k-PLE RIEMANN ZETA FUNCTION

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Abstract. In this paper, we construct a p-adic analogue of multiple Riemann zeta values and express their values at non-positive integers. In particular, we obtain a new congruence of the higher order Frobenius-Euler numbers and the Kummer congruences for the Bernoulli numbers as a corollary.

1. Introduction

Let ε be a root of unity of order relatively prime with p and $\varepsilon \neq 1$. We consider the Frobenius-Euler numbers $H_m(\varepsilon)$ defined by

$$\frac{\varepsilon - 1}{\varepsilon e^t - 1} = \sum_{m=0}^{\infty} H_m(\varepsilon)\frac{t^m}{m!}, \tag{1.1}$$

which can be written symbolically as $e^{H(\varepsilon)t} = (\varepsilon - 1)/(\varepsilon e^t - 1)$, interpreted to mean that $(H(\varepsilon))^m$ must be replaced by $H_m(\varepsilon)$ when expand on the left (cf. [9, 13]). This relation can also be written $\varepsilon e^{(H(\varepsilon)+1)t} - e^{H(\varepsilon)t} = \varepsilon - 1$, or, if we equate powers of t,

$$H_0(\varepsilon) = 1, \quad \varepsilon(H(\varepsilon)+1)^m - H_m(\varepsilon) = 0 \quad \text{if } m \geq 1, \tag{1.2}$$

where again we must first expand and then replace $(H(\varepsilon))^i$ by $H_i(\varepsilon)$. We note that

$$H_m(-1) = E_m, \tag{1.3}$$

where E_m denotes the so-called Euler numbers (cf. [8, 9]). The Frobenius-Euler polynomials $H_m(x,\varepsilon)$ are defined by

$$H_m(x,\varepsilon) = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} H_i(\varepsilon). \tag{1.4}$$

Received September 27, 2010.
2010 *Mathematics Subject Classification.* 11B68, 11S80.
Key words and phrases. p-adic analogues, higher order Frobenius-Euler numbers, k-ple zeta function, Kummer-type congruences.

We easily see that

$$H_{m-1}(-1) = \frac{2}{m}(1-2^m)B_m, \quad m \geq 1. \tag{1.5}$$

Here the Bernoulli numbers are defined by

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{1.6}$$

The Bernoulli polynomials $B_m(x)$ are also defined by $B_m(x) = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} B_i$.

Among many properties of Bernoulli numbers the Kummer congruences for Bernoulli numbers are widely known [2, 5, 19, 20]. Kummer congruences of Bernoulli numbers were first known to us by Kummer [12] a century ago, but their interpretation in terms of p-adic interpolation of the Riemann zeta function was only discovered in 1964 by Kubota and Leopoldt [11]. In 1910, Frobenius [4] gave a generalization of the Kummer congruence. Vandiver [19] obtained the complementary congruences, which were extended by Carlitz [2] in many directions. Congruences for higher order Bernoulli numbers have been studied by many authors, Adelberg [1], Carlitz [3], Howard [5], etc.

In [13], Osipov's congruences are the generalization of the Kummer congruences for ordinary Bernoulli numbers. He also obtained the Witt's formula of the numbers $H_m(\varepsilon)$, which of the similar kinds are given in [6, 8, 10, 11, 14, 15, 16, 17, 18]. Recently, Kim and Lee [9] obtained some interesting identities related to the Frobenius-Euler polynomials $H_m(x,\varepsilon)$ by using the ordinary fermionic p-adic invariant integral on $\mathbb{Z}_p$.

In this paper we construct a p-adic analogue of k-ple Riemann zeta function and express their values at non-positive integers. Also, we obtain a new congruence of the higher order Frobenius-Euler numbers and the Kummer congruences for the Bernoulli numbers as a corollary.

2. The values of k-ple Riemann zeta function at non-positive integers

Let ε be roots of unity of order relatively prime with p and $\varepsilon \neq 1$. Then the higher order Frobenius-Euler numbers are defined by means of the following generating function

$$g_\varepsilon(t) = \left(\frac{1-\varepsilon}{1-\varepsilon e^t}\right)^k = \sum_{m=0}^{\infty} H_m^{(k)}(\varepsilon)\frac{t^m}{m!}. \tag{2.1}$$

The higher order Frobenius-Euler polynomials are also defined by means of the following generating function

$$g_\varepsilon(x,t) = g_\varepsilon(t)e^{xt} = \sum_{m=0}^{\infty} H_m^{(k)}(x,\varepsilon)\frac{t^m}{m!}. \tag{2.2}$$

Setting $x = 0$ in (2.2), $H_m^{(k)}(0, \varepsilon) = H_m^{(k)}(\varepsilon)$. If $k = 1$, it is less well known that the explicit representations for the Frobenius-Euler numbers and polynomials, complementing those given in [8, 9, 14]. Setting $\varepsilon = -1$ in (2.2), $H_m^{(k)}(x, -1) = E_m^{(k)}(x)$ are called the higher order Euler polynomials; setting $k = 1$ and $\varepsilon = -1$ in (2.2), $H_m^{(1)}(x, -1) = E_m(x)$ are called the classical Euler polynomials.

Let x be a positive real number and let $|\varepsilon| \leq 1$. The k-ple Riemann zeta function $\zeta_k(s, x, \varepsilon)$ is defined by

$$\zeta_k(s, x, \varepsilon) = \sum_{n_1, \ldots, n_k = 0}^{\infty} \frac{\varepsilon^{n_1 + \cdots + n_k}}{(x + n_1 + \cdots + n_k)^s}. \tag{2.3}$$

In practice, the k-ple Riemann zeta function $\zeta_k(s, x, \varepsilon)$ for $s = 0, -1, -2, \ldots$ are of particular interest. We shall discuss these matters as follows.

The k-ple Riemann zeta function $\zeta_k(s, x, \varepsilon)$ is expressed as an integral,

$$\Gamma(s)\zeta_k(s, x, \varepsilon) = \int_0^{\infty} \frac{e^{-xt} t^{s-1}}{(1 - \varepsilon e^{-t})^k} dt, \tag{2.4}$$

where $\Gamma(s)$ is the gamma function, which satisfies $\Gamma(s + 1) = s\Gamma(s), \Gamma(1) = 1$, so that, in particular, $\Gamma(m) = (m - 1)!$ for positive integers m. Let C denote the contour which starts from $+\infty$, runs on the real axis, encircling the origin once counter-clockwise on the circle of small radius with the center at 0, runs the real axis and returns to $+\infty$. Since

$$\int_C \frac{e^{-xz} z^{s-1}}{(1 - \varepsilon e^{-z})^k} dz = (e^{2\pi i s} - 1) \int_0^{\infty} \frac{e^{-xt} t^{s-1}}{(1 - \varepsilon e^{-t})^k} dt,$$

we have

$$\zeta_k(s, x, \varepsilon) = \frac{e^{-\pi i s}\Gamma(1 - s)}{2\pi i} \int_C \frac{e^{-xz} z^{s-1}}{(1 - \varepsilon e^{-z})^k} dz. \tag{2.5}$$

This is the main virtue to obtain a contour integral representation for an analytic function. In particular, we see that $\zeta_k(s, x, \varepsilon)$ can be continued analytically to the whole s-plane (cf. [16, 20]). Furthermore, by (2.2) and (2.4), sufficiently large N we have

$$\begin{aligned} (1 - \varepsilon)^k \zeta_k(s, x, \varepsilon) = & \sum_{m=0}^{N} \frac{H_m^{(k)}(x, \varepsilon)}{m!\Gamma(s)} \frac{(-1)^m}{s + m} + \frac{1}{\Gamma(s)} H_N(s) \\ & + \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} g_\varepsilon(x, -t) dt, \end{aligned} \tag{2.6}$$

where $H_N(s)$ is entire. For an integer $m \geq 0$, we have

$$(1 - \varepsilon)^k \lim_{s \to -m} (s + m)\Gamma(s)\zeta_k(s, x, \varepsilon) = H_m^{(k)}(x, \varepsilon) \frac{(-1)^m}{m!}. \tag{2.7}$$

If $m \geq 0$, we have $\lim_{s \to -m}(s + m)\Gamma(s) = (-1)^m m!$ and thus we obtain the following lemma.

Lemma 2.1. *For $m \geq 0$ and $\varepsilon \neq 1$,*

$$\zeta_k(-m, x, \varepsilon) = \frac{H_m^{(k)}(x, \varepsilon)}{(1-\varepsilon)^k}.$$

Define

$$\tilde{\zeta}_k(s, x, \varepsilon) = \sum_{\substack{n_1, \ldots, n_k = 0 \\ p \nmid (n_1 + \cdots + n_k)}}^{\infty} \frac{\varepsilon^{n_1 + \cdots + n_k}}{(x + n_1 + \cdots + n_k)^s}. \tag{2.8}$$

For the special case of $\tilde{\zeta}_k(s, x, \varepsilon)$, i.e., when $s = 0, -1, -2, \ldots$, it is clear that from (2.3) and (2.8)

$$\begin{aligned}
\tilde{\zeta}_k(-m, x, \varepsilon) &= \zeta_k(-m, x, \varepsilon) - \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \sum_{n_1, \ldots, n_k = 0}^{\infty} \frac{\varepsilon^{|a| + p(n_1 + \cdots + n_k)}}{(x + |a| + p(n_1 + \cdots + n_k))^{-m}} \\
&= \zeta_k(-m, x, \varepsilon) - p^m \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} \zeta_k\left(-m, \frac{x + |a|}{p}, \varepsilon^p\right),
\end{aligned}$$

where $m \geq 0$ and $|a| = a_1 + \cdots + a_k$ (cf. [10]). It follows from this and Lemma 2.1 that

$$\begin{aligned}
&H_m^{(k)}(x, \varepsilon) - p^m \left(\frac{1}{[p]_\varepsilon}\right)^k \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x + |a|}{p}, \varepsilon^p\right) \\
&= (1-\varepsilon)^k \left(\zeta_k(-m, x, \varepsilon) - p^m \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} \zeta_k\left(-m, \frac{x + |a|}{p}, \varepsilon^p\right)\right) \\
&= (1-\varepsilon)^k \tilde{\zeta}_k(-m, x, \varepsilon).
\end{aligned}$$

Lemma 2.2. *Let $m \geq 0$ and $|a| = a_1 + \cdots + a_k$. Then*

$$\tilde{\zeta}_k(-m, x, \varepsilon) = \frac{1}{(1-\varepsilon)^k} \left(H_m^{(k)}(x, \varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x + |a|}{p}, \varepsilon^p\right)\right).$$

3. p-adic k-ple Riemann zeta function and Kummer-type congruences

In this section, let p be an odd prime number. The symbol $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ denote the rings of p-adic integers, the field of p-adic numbers and the field of p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p = 1/p$.

We denote two particular subrings of $\mathbb{C}_p$ in the following manner

$$\mathfrak{o}_p = \{s \in \mathbb{C}_p \mid |s|_p \leq 1\}, \quad \mathfrak{m}_p = \{s \in \mathbb{C}_p \mid |s|_p < 1\}.$$

Then $\mathfrak{m}_p$ is a maximal ideal of $\mathfrak{o}_p$. If $s \in \mathbb{C}_p$ such that $|s|_p \leq |p|_p^r$, where $r \in \mathbb{Q}$, then $s \in p^r \mathfrak{o}_p$, and so we shall also write this as $s \equiv 0 \pmod{p^r \mathfrak{o}_p}$ (cf. [6, 10, 20]).

Note that the two fields $\mathbb{C}$ and $\mathbb{C}_p$ are algebraically isomorphic, and any one of the two can be embedded in the other.

We begin with the following result.

Lemma 3.1. *Let $\varepsilon^r = 1, \varepsilon \neq 1$ and $(r,p) = 1$. Then there exists h such that $r \mid (p^h - 1)$, and*

$$H_0(\varepsilon) = 1, \quad \lim_{n\to\infty} \sum_{a=0}^{p^{hn}-1} a^m \varepsilon^a = H_m(\varepsilon), \quad m \geq 1.$$

Proof. Put $h = \varphi(r)$, where φ is the Euler function. Then $p^{\varphi(r)} \equiv 1 \pmod r$ since $(r,p) = 1$. This gives $p^{\varphi(r)n} \equiv 1 \pmod r, n \geq 0$ and so $r \mid (p^{\varphi(r)n} - 1)$. That is $\varepsilon^{p^{hn}} = \varepsilon$. Thus we have

$$\begin{aligned}\sum_{m=0}^{\infty} \left(\lim_{n\to\infty} \sum_{a=0}^{p^{hn}-1} a^m \varepsilon^a \right) \frac{t^m}{m!} &= \lim_{n\to\infty} \sum_{a=0}^{p^{hn}-1} e^{at}\varepsilon^a = \lim_{n\to\infty} \frac{\varepsilon^{p^{hn}} e^{tp^{hn}} - 1}{\varepsilon e^t - 1} \\ &= \frac{\varepsilon - 1}{\varepsilon e^t - 1} = \sum_{m=0}^{\infty} H_m(\varepsilon) \frac{t^m}{m!},\end{aligned}$$

where $|t|_p < p^{-1/(p-1)}$. The result follows at once. □

Let r be a positive integer prime to p, and $\varepsilon \in \mathbb{C}_p$ a r-th root of unity different from 1. Let $f : \mathbb{Z}_p^k \to \mathbb{C}_p$ be any continuous function and let $a = (a_1, \ldots, a_k)$ be a variable on $\mathbb{Z}_p^k$. We define the p-adic integration of f on $\mathbb{Z}_p^k$, if it exists, by the formula

$$\int_{\mathbb{Z}_p^k} f(a) d\mu_\varepsilon(a) = \lim_{\substack{n_1\to\infty \\ \cdots \\ n_k\to\infty}} \sum_{a_1=0}^{p^{hn_1}-1} \cdots \sum_{a_k=0}^{p^{hn_k}-1} f(a_1, \ldots, a_k) \varepsilon^{a_1} \cdots \varepsilon^{a_k}, \tag{3.1}$$

where h is a positive integer such that $r \mid (p^h - 1)$ (cf. [13]).

Lemma 3.2. *For integer $m \geq 0$ and $x \in \mathbb{C}_p$,*

$$H_m^{(k)}(x, \varepsilon) = \int_{\mathbb{Z}_p^k} (x + |a|)^m d\mu_\varepsilon(a),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. Note that

$$\begin{aligned}\int_{\mathbb{Z}_p^k} e^{t(x+|a|)} d\mu_\varepsilon(a) &= \lim_{\substack{n_1\to\infty\\ \cdots \\ n_k\to\infty}} \sum_{a_1=0}^{p^{hn_1}-1} \cdots \sum_{a_k=0}^{p^{hn_k}-1} e^{t(x+a_1+\cdots+a_k)} \varepsilon^{a_1}\cdots\varepsilon^{a_k} \\ &= e^{tx}\prod_{i=1}^{k}\left(\lim_{n_i\to\infty}\frac{1-\varepsilon^{p^{hn_i}} e^{tp^{hn_i}}}{1-\varepsilon e^t}\right) = g_\varepsilon(x,t)\end{aligned}$$

(cf. [10]). Taking the coefficient of the terms $t^m/m!$ in the above formula, we obtain the lemma. □

Put $|a| = a_1 + \cdots + a_n$. Let $a = (a_1, \ldots, a_k)$ be a variable on $\mathbb{Z}_p^k$ and let $\mathbb{Z}_p^\times$ be the group of p-adic units. It is easy to see that

$$\int_{\substack{\mathbb{Z}_p^k\\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) = \int_{\mathbb{Z}_p^k} (x+|a|)^m d\mu_\varepsilon(a) - \int_{\substack{\mathbb{Z}_p^k\\ |a|\in p\mathbb{Z}_p}} (x+|a|)^m d\mu_\varepsilon(a) \tag{3.2}$$

(cf. [10]). We use the notation

$$[n]_\varepsilon = \frac{1-\varepsilon^n}{1-\varepsilon}.$$

Now, we need to compute (3.2). The following lemma deals with the second integral in (3.2).

Lemma 3.3. *For integer $m \geq 0$ and $x \in \mathbb{C}_p$,*

$$\int_{\substack{\mathbb{Z}_p^k\\ |a|\in p\mathbb{Z}_p}} (x+|a|)^m d\mu_\varepsilon(a) = \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0\\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. Note that

$$\begin{aligned}&\int_{\substack{\mathbb{Z}_p^k\\ |a|\in p\mathbb{Z}_p}} e^{t(x+|a|)} d\mu_\varepsilon(a) \\ &= e^{tx}\lim_{n_1\to\infty}\cdots\lim_{n_k\to\infty} \sum_{\substack{a_1,\ldots,a_k=0\\ p\nmid|a|}}^{p-1} \sum_{b_1=0}^{p^{hn_1-1}-1} \cdots \sum_{b_k=0}^{p^{hn_k-1}-1} \prod_{i=1}^{k} (\varepsilon e^t)^{a_i+pb_i} \\ &= e^{tx}\lim_{n_1\to\infty}\cdots\lim_{n_k\to\infty} \sum_{\substack{a_1,\ldots,a_k=0\\ p\nmid|a|}}^{p-1} \prod_{i=1}^{k}(\varepsilon a^t)^{a_i} \sum_{b_1=0}^{p^{hn_1-1}-1} \cdots \sum_{b_1=0}^{p^{hn_k-1}-1} \prod_{i=1}^{k}(\varepsilon e^t)^{pb_i} \\ &= \sum_{\substack{a_1,\ldots,a_k=0\\ p\nmid|a|}}^{p-1} \varepsilon^{a_1+\cdots+a_k} e^{t(x+a_1+\cdots+a_k)} \lim_{n_1\to\infty}\cdots\lim_{n_k\to\infty} \prod_{i=1}^{k}\left(\frac{1-\varepsilon e^{tp^{hn_i}}}{1-\varepsilon^p e^{tp}}\right)\end{aligned}$$

$$= \frac{1}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{a_1+\cdots+a_k} g_{\varepsilon^p}(tp) e^{t(x+a_1+\cdots+a_k)}.$$

Taking the coefficient of the terms $t^m/m!$ in the above formula, we obtain the lemma. □

Lemma 3.4. *For integer $m \geq 0$ and $x \in \mathbb{C}_p$,*

$$\int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) = H_m^{(k)}(x,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. By (3.2), Lemmas 3.2 and 3.3 we obtain the desired identity. □

Lemma 3.5. *Let $x \in \mathfrak{m}_p$. The function*

$$-m \longmapsto H_m^{(k)}(x,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right)$$

admits a continuation from the dense subset $\{0,-1,\ldots\} \subset \mathbb{Z}_p$ to a continuous function

$$\zeta_{p,k}(\cdot, x, \varepsilon) : \mathbb{Z}_p \to \mathbb{C}_p$$

and

$$\zeta_{p,k}(s,x,\varepsilon) = \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^{-s} d\mu_\varepsilon(a),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. Let $|a| \in \mathbb{Z}_p^\times, x \in \mathfrak{m}_p$ and let $m \equiv m' \pmod{(p-1)p^n}$. It is easy to see that $(x+|a|)^m \equiv (x+|a|)^{m'} \pmod{p^{n+1}\mathfrak{o}_p}$. Therefore we have

$$\int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) \equiv \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^{m'} d\mu_\varepsilon(a) \pmod{p^{n+1}\mathfrak{o}_p} \tag{3.3}$$

and they would also belong to a continuous p-adic function on $\mathbb{Z}_p$. The result now follows from Lemma 3.4. □

If $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbb{Z}_p^\times$, $a + pt \equiv a \pmod{p\mathfrak{o}_p}$. Thus we define

$$\omega(a+pt) = \omega(a)$$

for these values of t and the Teichmüller character ω. We also define

$$\langle a+pt \rangle = \omega^{-1}(a)(a+pt)$$

for $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$ (cf. [16, 20]). We define a function $\zeta_{p,k}(s,t,\varepsilon)$ on $\mathbb{Z}_p$ by

$$\zeta_{p,k}(s,t,\varepsilon) = \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} \langle |a| + pt \rangle^{-s} d\mu_\varepsilon(a), \tag{3.4}$$

where $|a| = a_1 + \cdots + a_k$ and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$.

Theorem 3.6 (p-adic k-ple Riemann zeta function). *For $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, the function $\zeta_{p,k}(s,t,\varepsilon)$ is analytic on $\mathbb{Z}_p$ and*

$$\zeta_{p,k}(s,t,\varepsilon) = \sum_{n=0}^{\infty} \binom{-s}{n} \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a)$$

holds, which interpolates $(1-\varepsilon)^k \tilde{\zeta}_k(-m, pt, \varepsilon)$ in the sense that

$$\zeta_{p,k}(-m,t,\varepsilon) = H_m^{(k)}(pt,\varepsilon) - p^m \left(\frac{1}{[p]_\varepsilon}\right)^k \sum_{\substack{a_1,\ldots,a_k=0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(t + \frac{|a|}{p}, \varepsilon^p\right)$$

for integers $m \geq 0$ with $m \equiv 0 \pmod{p-1}$ and $|a| = a_1 + \cdots + a_k$.

Proof. From Lemma 3.5, $\zeta_{p,k}(-s,t,\varepsilon)$ can be written uniquely as the Mahler expansion (cf. [20])

$$\zeta_{p,k}(-s,t,\varepsilon) = \sum_{n=0}^{\infty} a_n \binom{s}{n}, \quad a_n = \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a)$$

and

$$\begin{aligned} |a_n|_p &= \left| \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a) \right|_p \\ &\leq \sup_{\substack{a \in \mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} |\langle |a| + pt \rangle - 1|_p^n \\ &= p^{-n} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Note that the coefficients a_n are given by

$$a_n = \Delta^n \zeta_{p,k}(-s,t,\varepsilon)|_{s=0},$$

where $\Delta f(x) = f(x+1) - f(x)$. Moreover we have

$$\frac{1}{n!} a_n \to 0 \quad \text{as} \quad n \to \infty,$$

so that $\zeta_{p,k}(s,t,\varepsilon)$ is analytic. Therefore the result follows from Lemma 2.2 and Lemma 3.5. □

From Lemma 3.4 and (3.3), we also have:

Corollary 3.7. *Let* $m \equiv m' \pmod{p^n(p-1)}$ *and let* $t \in \mathbb{C}_p$ *such that* $|t|_p \le 1$. *Then*

$$\begin{aligned} & H_m^{(k)}(pt,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(t+\frac{|a|}{p}, \varepsilon^p\right) \\ \equiv\ & H_{m'}^{(k)}(pt,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_{m'}^{(k)}\left(t+\frac{|a|}{p}, \varepsilon^p\right) \pmod{p^{n+1}\mathfrak{o}_p}. \end{aligned}$$

In particular if $t = 0$ and $k = 1$, we an rewrite Corollary 3.7 as

$$H_m(\varepsilon) - \frac{p^m}{[p]_\varepsilon} H_m(\varepsilon^p) \equiv H_{m'}(\varepsilon) - \frac{p^{m'}}{[p]_\varepsilon} H_{m'}(\varepsilon^p) \pmod{p^{n+1}\mathfrak{o}_p}, \tag{3.5}$$

which is the same as (23) in [13]. If $\varepsilon = -1$ in (3.5), then we have the following corollary.

Corollary 3.8. *If* $m \equiv m' \pmod{p^n(p-1)}$, *then*

$$(1-p^m)H_m(-1) \equiv (1-p^{m'})H_{m'}(-1) \pmod{p^{n+1}\mathbb{Z}_p}.$$

By (1.5) and Corollary 3.8, it is easy to see that

$$(1-p^m)(1-2^{m+1})\frac{B_{m+1}}{m+1} \equiv (1-p^{m'})(1-2^{m'+1})\frac{B_{m'+1}}{m'+1} \pmod{p^{n+1}\mathbb{Z}_p}. \tag{3.6}$$

If we further assume that $m+1 \not\equiv 0 \pmod{p-1}$, then we have $1/(1-2^{m+1}) \equiv 1/(1-2^{m'+1}) \pmod{p^{n+1}\mathbb{Z}_p}$. Multiplying these two congruences, we obtain the Kummer congruences for the Bernoulli numbers (see [13, 20]):

Corollary 3.9 (Kummer congruences)**.** *If* $m+1 \not\equiv 0 \pmod{p-1}$ *and if* $m \equiv m' \pmod{p^n(p-1)}$, *then*

$$(1-p^m)\frac{B_{m+1}}{m+1} \equiv (1-p^{m'})\frac{B_{m'+1}}{m'+1} \pmod{p^{n+1}\mathbb{Z}_p}.$$

Acknowledgements. This work was supported by a Kyungnam University Foundation grant in 2010.

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